

Causal band-limitness and predictability criteria for one-sided sequences

Nikolai Dokuchaev

Submitted: August 25, 2014. Revised: September 23, 2014

Abstract

The paper studies frequency characteristics and predictability of real sequences, i.e., discrete time processes in deterministic setting. We consider band-limitness and predictability of one-sided sequences. We establish predictability of some classes of sequences, and we suggest predictors represented by causal convolution sums over past times.

Keywords: Bandlimited sequences, one-sided sequences, discrete time systems, prediction, Szegő-Kolmogorov Theorem.

MSC 2010 classification : 42A38, 93E10, 62M15, 42B30

1 Introduction

The paper studies frequency characteristics and predictability of real sequences, i.e., discrete time processes in deterministic setting. It is well known that certain restrictions on the frequency distribution can ensure some opportunities for prediction and interpolation of the processes. For continuous time processes, the classical result is the Nyquist-Shannon-Kotelnikov interpolation theorem for the sampling of continuous time band-limited processes. These processes are analytic; they allow a unique extrapolation from any interval and are uniquely defined by their past. A similar result was obtained for the processes with the exponential decay of energy on the higher frequencies that are not necessary band-limited [4]. Predictability based on sampling and the Nyquist-Shannon-Kotelnikov theorem was discussed, e.g., in [10, 11, 13, 14, 15, 16, 17, 18]. These works considered continuous-time band-limited stochastic processes; the corresponding predictors were constructed for the setting where the shape of

The author is with Department of Mathematics and Statistics, Curtin University, GPO Box U1987, Perth, Western Australia, 6845 (email N.Dokuchaev@curtin.edu.au). This work was supported by ARC grant of Australia DP120100928 to the author.

the spectral representation is supposed to be known. For discrete time processes, the predictability can also be achieved given some properties of spectral representations.

For discrete time processes or sequences, it is not obvious how to define an analog of the continuous time analyticity. So far, there is a criterion of predictability in the frequency domain setting given by the classical Szegő-Kolmogorov Theorem for stochastic Gaussian stationary discrete time processes. This theorem says that the optimal prediction error is zero if the spectral density ϕ is such that

$$\int_{-\pi}^{\pi} \log \phi(e^{i\omega}) d\omega = -\infty; \quad (1)$$

see [12, 21, 22] and recent literature reviews in [1, 20]. This means that a stationary Gaussian process is predictable if its spectral density is vanishing on a part of the unit circle $\mathbb{T} = \{z \in \mathbf{C} : |z| = 1\}$, i.e., if the process is band-limited in this sense. This result was expanded on more general stable stochastic processes allowing spectral representations with spectral density via processes with independent increments (see, e.g., [2]). In [4, 5, 6, 7, 8], this problem was readdressed in the deterministic setting, and some predictors were suggested for band-limited sequences, i.e., for sequences for which Z-transform vanishing on a part of \mathbb{T} . However, it appears that the framework of two-sided sequences required for detecting of the band-limitness via Z-transforms are not always convenient to use. For example, consider a situation where the data is collected dynamically during a prolonged time interval. Application of the two-sided Z-transform requires to select some past time at the middle of the time interval of the observations as the zero point for a model of the two-sided sequence; this could be inconvenient. Therefore, it could be more convenient to analyze one-sided sequences rather than two-sided sequences required for detecting of the band-limitness via Z-transforms. This leads to the analysis of the one-sided sequences directed backward to the past. However, the straightforward application of the one-sided Z-transform to the historical data considered as an one-sided sequence directed to the past does not generate Z-transform vanishing on a part of the unit circle even for a band-limited underlying sequence. To overcome this, we use sine and cosine versions of Z-transforms; it appears that they allow to detect band-limitness in one-sided sequences. This gives a possibility to establish predictability of certain classes of one-sided sequences. Following [4, 5, 6], we suggest predictors represented by causal convolution sums over past times representing historical observations.

2 Definitions and the main results

We denote by \mathbf{Z} the set of all integers.

For $\tau \in \mathbf{Z} \cup \{+\infty\}$ and $\theta < \tau$, we denote by $\ell_r(\theta, \tau)$ a Banach space of sequences $x = \{x(t)\}_{\theta-1 < t < \tau+1} \subset \mathbf{R}$, with the norm $\|x\|_{\ell_r(\theta, \tau)} = (\sum_{t=\theta}^{\tau} |x(t)|^r)^{1/r} < +\infty$ for $r \in [1, \infty)$ or $\|x\|_{\ell_\infty(\theta, \tau)} = \sup_{t: \theta-1 < t < \tau+1} |x(t)| < +\infty$ for $r = +\infty$; the cases where $\theta = -\infty$ or $\tau = +\infty$ are

not excluded. As usual, we assume that all sequences with the finite norm of this kind are included in the corresponding space.

For brevity, we will use the notations $\ell_r = \ell_r(-\infty, \infty)$, and $\ell_r^- = \ell_r(-\infty, 0)$.

2.1 The classical Z-transform and band-limitness

For $x \in \ell_1$ or $x \in \ell_2$, we denote by $X = \mathcal{Z}x$ its Z-transform defined as

$$X(z) = \sum_{t=-\infty}^{\infty} x(t)z^{-t}, \quad z \in \mathbf{C}.$$

Respectively, the inverse $x = \mathcal{Z}^{-1}X$ is defined as

$$x(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{i\omega}) e^{i\omega t} d\omega, \quad t = 0, \pm 1, \pm 2, \dots$$

Let $\mathbb{T} \triangleq \{z \in \mathbf{C} : |z| = 1\}$, and let $\mathbb{T}^+ \triangleq \{z \in \mathbf{C} : |z| = 1, \operatorname{Im} z \geq 0\}$. If $x \in \ell_2$, then $X|_{\mathbb{T}}$ is defined as an element of $L_2(\mathbb{T})$. Let the mapping $\mathcal{Z}^+ : \ell_2 \rightarrow L_2(\mathbb{T}^+)$ be defined as $\mathcal{Z}^+x = (\mathcal{Z}x)|_{\mathbb{T}^+}$. It can be noted that $\overline{X(e^{i\omega})} = X(e^{-i\omega})$ for $X = \mathcal{Z}x$, $x \in \ell_2$, and the mapping $\mathcal{Z}^+ : \ell_2 \rightarrow L_2(\mathbb{T}^+)$ is a continuous bijection between ℓ_2 and $L_2(\mathbb{T}^+)$.

Definition 1 *We will call a two-sided sequence $x \in \ell_2$ band-limited if there exists $\Omega \in [0, \pi)$ such that $X(e^{i\omega}) = 0$ for $\omega \in [-\pi, \Omega) \cup (\Omega, \pi]$, where $X = \mathcal{Z}x$.*

It is known that two-sided band-limited infinite sequences are predictable in a certain sense (see Theorem 1 in [5] and Theorem 1 in [6]).

As was mentioned in Section 1, for many practical applications, it is inconvenient to estimate frequency characteristics of two-sided infinite sequences. On the other hand, the unilateral Z-transform calculated for a one-sided part of a band-limited process does not show its band-limitness. More precisely, if we consider a band-limited sequence $x \in \ell_2$, then the process $x_\theta(t) = x(t)\mathbb{I}_{(-\infty, t]}(t)$ cannot be band-limited for a given $\theta \in \mathbf{Z}$, and $\ln |(\mathcal{Z}x_\theta)(e^{i\omega})| \in L_1(-\pi, \pi)$, there are constraints on the degeneracy of $\mathcal{Z}x_\theta$ on \mathbb{T} .

2.2 Sine and cosine Z-transforms for one-sided sequences

We suggest below sine and cosine modifications of Z-transform oriented on applications to the one-sided sequences.

For $x \in \ell_1^-$ or $x \in \ell_2^-$, we introduce transforms $\xi_1 = \Xi_1 x \in L_2([0, \pi], \mathbf{R})$ and $\xi_2 = \Xi_2 x \in L_2([0, \pi], \mathbf{R}) \times \mathbf{R}$ defined as

$$\begin{aligned}\xi_1(\omega) &= 2 \sum_{t=-\infty}^{-1} \cos(\omega t) x(t) + x(0), \\ \xi_2(\omega) &= \left(2 \sum_{t=-\infty}^{-1} \sin(-\omega t) x(t), x(0) \right),\end{aligned}$$

where $\omega \in [0, \pi]$.

Lemma 1 *The mappings $\Xi_1 : \ell_2^- \rightarrow L_2([0, \pi], \mathbf{R})$ and $\Xi_2 : \ell_2^- \rightarrow L_2([0, \pi], \mathbf{R}) \times \mathbf{R}$, are continuous bijections, and the corresponding inverse mappings are also a continuous bijections such that $x_k = \Xi_k^{-1} \xi_k$ are defined as*

$$\begin{aligned}x_1(t) &= \frac{1}{\pi} \int_0^\pi \xi_1(\omega) \cos(\omega t) d\omega, \quad t = 0, -1, -2, \dots, \\ x_2(t) &= \frac{1}{\pi} \int_0^\pi \xi_2'(\omega) \sin(-\omega t) d\omega, \quad t = -1, -2, \dots \\ x_2(0) &= \xi_2'', \quad \text{where } \xi_2 = (\xi_2'(\omega), \xi_2'').\end{aligned} \tag{2}$$

2.2.1 Application to band-limitness and predictability

To justify the introduction of the new transforms Ξ_k , we demonstrate below that, similarly to the band-limitness defined via the Z-transform for two-sided infinite sequences, it is possible to define a detectable analog of the band-limitness for the one-sided infinite sequences. Moreover, we will show that the presence of this new band-limitness also leads to a predictability.

Definition 2 *Let $r \in \{1, 2\}$. We will call a one-sided sequence $x \in \ell_r^-$ causally band-limited (or left band-limited) if there exists $\Omega \in [0, \pi)$ and $x' \in \ell_r(1, +\infty)$ such that $X(e^{i\omega}) = 0$ for $|\omega| > \Omega$, where $X = \mathcal{Z}\bar{x}$, and where $\bar{x} \in \ell_r$ is such that $\bar{x}(t) = x(t)$ for $t \leq 0$ and $\bar{x}(t) = x'(t)$ for $t > 0$.*

In principle, it is possible to verify if x the conditions of this definition holds with a given $\Omega \in [0, \pi)$. In particular, it can be done using Theorem 1 from [7, 8] via solution of a infinity dimensional quadratic optimization problem which is a computationally challenging procedure. Moreover, this can be done for each potentially acceptable Ω separately.

We suggest below a more convenient sufficient conditions of band-limitness.

Theorem 1 *A sequence $x \in \ell_2^-$ is causally band-limited if there exists $\Omega \in [0, \pi)$ such that at least one of the following two condition holds:*

- (i) *There exists $a \in \mathbf{R}$ such that $\xi_1(\omega) = a$ for $\omega \in (\Omega, \pi]$, where $\xi_1 = \Xi_1 x$;*

(ii) $\xi_2'(\omega) = 0$ for $\omega \in (\Omega, \pi]$, where $\xi_2 = (\xi_2'(\omega), \xi_2'') = \Xi_2 x$.

It can be noted that a sequence can be causally band-limited even if the transform $\xi_k(\omega)$ are separated from zero on $[0, \pi]$; this feature makes causal band-limitness different from the band-limitness defined for the two-sided sequences via degeneracy of the Z-transform.

Remark 1 *The conditions on ξ_1 and ξ_2' required in Theorem 1 cannot be satisfied simultaneously.*

3 Predicability of causally band-limited processes and more general processes

Let $D \triangleq \{z \in \mathbf{C} : |z| \leq 1\}$, $D^c = \mathbf{C} \setminus D$. For $r \in [1, +\infty]$, let H^r be the Hardy space of functions that are holomorphic on D^c including the point at infinity (see, e.g., [9]). Note that Z-transform defines a bijection between the sequences from ℓ_2^+ and the restrictions (i.e., traces) $X|_{\mathbb{T}}$ of the functions from H^2 on \mathbb{T} such that $\overline{X(e^{i\omega})} = X(e^{-i\omega})$.

Definition 3 *Let $\widehat{\mathcal{K}}$ be the class of functions $\widehat{k} : \ell_\infty^+ \rightarrow \mathbf{R}$ such that $\widehat{k}(t) = 0$ for $t < 0$ and such that $\widehat{K}(\cdot) = \mathcal{Z}\widehat{k} \in H^\infty$.*

For $r \in [1, +\infty]$, let $s : \ell_r \rightarrow \ell_r$ be the shift operator defined as $(sx)(t) = x(t+1)$.

Definition 4 *Let $\mathcal{Y} \subset \ell_r^-$ be a class of one-sided sequences, $r \in [1, +\infty]$.*

(i) *We say that this class is unilaterally ℓ_r^- -predictable if there exists a sequence $\{\widehat{k}_m(\cdot)\}_{m=1}^{+\infty} \subset \widehat{\mathcal{K}}$ such that*

$$\|sx - \widehat{x}_m\|_{\ell_r(-\infty, -1)} \rightarrow 0 \quad \text{as } m \rightarrow +\infty \quad \forall x \in \mathcal{Y}.$$

$$\text{Here } \widehat{x}_m(t) \triangleq \sum_{s=-\infty}^t \widehat{k}_m(t-s)x(s).$$

(ii) *We say that the class \mathcal{Y} is unilaterally uniformly ℓ_r^- -predictable if, for any $\varepsilon > 0$, there exists $\widehat{k}(\cdot) \in \widehat{\mathcal{K}}$ such that*

$$\|sx - \widehat{x}\|_{\ell_r(-\infty, -1)} \leq \varepsilon \quad \forall x \in \mathcal{Y}.$$

$$\text{Here } \widehat{x}(t) \triangleq \sum_{s=-\infty}^t \widehat{k}(t-s)x(s).$$

For $r \in \{1, 2\}$, $\Omega \in [0, \Omega)$, let $\ell_{r,BL}^-(\Omega)$ be the set of all one-sided sequences causally band-limited $x \in \ell_2^-$ such that, for each $x \in \ell_{r,BL}^-(\Omega)$, the condition of Definition 2 are satisfied. For $d > 0$, let $\ell_{r,BL}^-(\Omega, d)$ be the set of all $x \in \ell_{r,BL}^-(\Omega)$ such that $\|x\|_{\ell_r^-} \leq d$. Let $\ell_{r,BL}^- = \cup_{\Omega \in [0, \pi)} \ell_{r,BL}^-(\Omega)$.

Theorem 2 (i) The class $\ell_{2,BL}^-$ is unilaterally ℓ_2^- -predictable.

(ii) For any $\Omega \in [0, \pi]$, $d > 0$, the class $\ell_{2,BL}^-(\Omega, d)$ is unilaterally uniformly ℓ_2^- -predictable.

(iii) Let $\mu > 1$ and $q > 1$ be given. A sequence of predicting kernels that ensures prediction required in (i) and (ii) can be constructed as the following: $\hat{k}(\cdot) = \hat{k}(\cdot, \gamma) = \mathcal{Z}^{-1} \hat{K}$, where

$$\hat{K}(z) = z \left(1 - \exp \left[-\frac{\gamma}{z + 1 - \gamma^{2\mu/(1-q)}} \right] \right). \quad (3)$$

Here $\gamma > 0$ is a parameter; the prediction error vanishes as $\gamma \rightarrow +\infty$.

The predicting kernels (3) were suggested in [5], Theorem 1 for two-sided sequences. They represent an extension on the discrete time setting of the construction introduced in [4] for continuous time processes.

Further, let some $q > 1$ be given. For $c > 0$ and $\omega \in [-\pi, \pi]$, set

$$h(\omega, c) = \exp \frac{c}{[(\cos(\omega) + 1)^2 + \sin^2(\omega)]^{q/2}}.$$

Let $\mathcal{W}(c)$ be the class of all sequences $x \in \ell_2^-$ such that, for each $x \in \ell_2^-$, at least one of the following conditions holds: either there exist $a \in \mathbf{R}$ such that

$$\text{ess sup}_{\omega \in [0, \pi]} |\xi_1(\omega) - a| h(\omega, c) < +\infty,$$

or

$$\text{ess sup}_{\omega \in [0, \pi]} |\xi_2'(\omega)| h(\omega, c) < +\infty.$$

Here $\xi_k = \Xi_k x$, $\xi_2 = (\xi_2'(\omega), \xi_2''(\omega))$. Let $\mathcal{W} = \cup_{c>0} \mathcal{W}(c)$.

Note that $h(\omega, c) \rightarrow +\infty$ as $\omega \rightarrow \pm\pi$ and that, for $x \in \mathcal{W}(c)$, either $\xi_1(\omega) - a$ or $\xi_2''(\omega)$ vanishing with a sufficient rate of decay as $\omega \rightarrow \pi$. In particular, $\ell_{2,BL}^- \subset \mathcal{W}$, i.e., the class \mathcal{W} includes all causally band-limited sequences.

Further, for $c > 0$ and $d > 0$, let $\mathcal{V}(c, d)$ be a class of processes $x \in \mathcal{W}(c)$ such that there exists $a \in \mathbf{R}$ such that

$$\min \left(\text{ess sup}_{\omega \in [0, \pi]} |(\xi_1(\omega) - a) h(\omega, c)|, \text{ess sup}_{\omega \in [0, \pi]} |\xi_2''(\omega) h(\omega, c)| \right) \leq d, \quad (4)$$

where $\xi_k = \Xi_k x$ and $\xi_2 = (\xi_2'(\omega), \xi_2''(\omega))$.

Theorem 3 Let either $r = 2$ or $r = +\infty$.

(i) The class \mathcal{W} is unilaterally ℓ_r^- -predictable.

(ii) For any given $c > 0$ be given and $d > 0$, the class $\mathcal{V}(c, d)$ is uniformly unilaterally ℓ_r -predictable.

(iii) A sequence of predicting kernels that ensures prediction required in (i) and (ii) can be constructed as defined in (3).

4 Proofs

Proof of Lemma 1. For $x \in \ell_2^-$, we define $\tilde{x}_k \in \ell_2$, $k = 1, 2$, such that $x_k(t) = x(t)$ for $t \leq 0$ and $\tilde{x}_1(t) = x(-t)$, $\tilde{x}_2(t) = -x(-t)$ for $t > 0$. For $\xi_k = \Xi_k x$ and $X_k = \mathcal{Z}\tilde{x}_k$, it can be verified that

$$X_1(e^{i\omega}) = \xi_1(|\omega|), \quad \omega \in [-\pi, \pi], \quad (5)$$

and, for $\xi_2 = (\xi_2'(\omega), \xi_2'')$,

$$\begin{aligned} X_2(e^{i\omega}) &= i\xi_2'(\omega) + \xi_2'', \quad \omega \in [0, \pi], \\ X_2(e^{i\omega}) &= -i\xi_2'(-\omega) + \xi_2'', \quad \omega \in [-\pi, 0). \end{aligned} \quad (6)$$

Since $\tilde{x}_k \in \ell_2$, it follows that $|X_k(e^{i\omega})| \in L_2([-\pi, \pi], \mathbf{R})$. Hence $\xi_k = \Xi_k x \in L_2([0, \pi], \mathbf{R})$, and the mappings $\Xi_1 : \ell_2^- \rightarrow L_2([0, \pi], \mathbf{R})$ and $\Xi_2 : \ell_2^- \rightarrow L_2([0, \pi], \mathbf{R}) \times \mathbf{R}$ are continuous.

Further, let $\xi_1 \in L_2([0, \pi], \mathbf{R})$ and $\xi_2 = (\xi_2'(\omega), \xi_2'') \in L_2([0, \pi], \mathbf{R}) \times \mathbf{R}$ be arbitrarily selected. Let us define mappings $X_k : \mathbb{T} \rightarrow \mathbf{R}$ defined by (5) and (6). Let $\tilde{x}_k = \mathcal{Z}^{-1}X_k$, i.e.,

$$\tilde{x}_k(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_k(e^{i\omega}) e^{i\omega t} d\omega, \quad t = 0, \pm 1, \pm 2, \dots$$

It can be verified immediately that $\tilde{x}_k(t)$ are real, $\tilde{x}_1(t) = \tilde{x}_1(-t)$, $\tilde{x}_2(t) = -\tilde{x}_2(-t)$, $t > 0$, and that (2) holds for $x_k \triangleq \hat{x}_k|_{\{t \leq 0\}}$. It follows that $\Xi_k x_k = \xi$. Hence the mappings $\Xi_1 : \ell_2^- \rightarrow L_2([0, \pi], \mathbf{R})$ and $\Xi_2 : \ell_2^- \rightarrow L_2([0, \pi], \mathbf{R}) \times \mathbf{R}$ are bijections. It is known that an inverse of a continuous bijection between Banach spaces is also continuous. This completes the proof of Lemma 1. \square

Proof of Theorem 1. It follows from the proof of Lemma 1 that if the condition of the theorem is satisfied that one of the processes \tilde{x}_k introduced in this proof is band-limited. This completes the proof of Theorem 1. \square

Proof of Remark 1. It follows from the proof of Theorem 1 that the conditions on ξ_1 and ξ_2' required in Theorem 1 cannot be satisfied simultaneously; this would contradict the uniqueness of the extrapolation of a band-limited processes established in [5, 6].

Theorem 2 follows immediately from Theorem 3. Therefore, it suffices to prove Theorem 3.

Proof of Theorem 3. Let $x \in \mathcal{W}$ and $\xi_k = \Xi_k x$, and let \tilde{x}_k and X_k be such as defined in the proof of Lemma 1 above; in particular, we assume that (5)-(6) hold.

Let us prove statement (i). Let $x \in \mathcal{W}(c, d)$ be given for some $c > 0$, $d > 0$. If the definition of $\mathcal{W}(c, d)$ holds for this x with the condition for $\xi_1(\omega)$, we set $k = 1$ and select $a \in \mathbf{R}$ to be the corresponding a . Otherwise, we set $k = 2$ and $a = x(0)$; this happens if the definition of $\mathcal{W}(c)$ holds for x with the condition for $\xi_2''(\omega)$ only. Consider a process $x^a(t)$ such that $x^a(t) = x(t)$ for $t < 0$ and $x^a(0) = x(0) - a$. Let $\xi_k^a = \Xi_k x^a$. By the definitions, it follows that $\text{ess sup}_{\omega \in [0, \pi]} |\zeta_k^a(\omega)| h(\omega, c) < +\infty$, where $\zeta_1^a = \xi_1$ and $\zeta_2^a = \xi_2''$, for $\xi_2 = (\xi_2'(\omega), \xi_2'')$.

Let $\tilde{x}_k^a(t) \in \ell_r$ be defined similarly to \tilde{x}_k in the proof of Lemma 1, with x replaced by x^a . By the definitions, \tilde{x}_k^a belongs to the class $\mathcal{X} \subset \ell_r$ used in Theorem 1 (i) [5]. By this theorem, the class \mathcal{X} is ℓ_r -predicable in the sense of Definition 2(i) [5]. Since $\tilde{x}_k^a(t) = x(t)$ for $x \in \mathcal{W}$, $t < 0$, it follows that the predicability of \mathcal{X} in the sense of Definition 2(i) [5] implies predictability of \mathcal{W} in the sense of Definition 4(i).

Let us prove statement (ii) following the same steps. Let $x \in \mathcal{V}(c, d)$. It suffices to consider $d = 1$ only. If the definition of $\mathcal{V}(c, 1)$ holds for this x with the minimum in (4) achieved for $\xi_1(\omega)$, we set $k = 1$ and select $a \in \mathbf{R}$ to be the corresponding a . Otherwise, we set $k = 2$ and $a = x(0)$; this happens if the minimum in (4) is achieved for $\xi_2''(\omega)$ only.

Let x^a , $\xi_k^a = \Xi_k x^a$, and $\tilde{x}_k^a(t) \in \ell_r$, be defined as in the proof for the statement (i) above.

By the definitions, \tilde{x}_k^a belong to the class $\mathcal{U}(c)$ defined in Theorem 1 (ii) [5]. The class $\mathcal{U}(c) \subset \ell_r$ is uniformly ℓ_r -predicable in the sense of Definition 2(ii) [5]. Since $\tilde{x}_k^a(t) = x(t)$ for $x \in \mathcal{V}(c, 1)$, $t < 0$, it follows that the predicability for $\mathcal{U}(c_0)$ in the sense of Definition 2(ii) [5] implies predictability in the sense of Definition 4(ii) for $\mathcal{V}(c, 1)$. This completes the proof of Theorem 3. \square

5 Discussion and future developments

1. It is possible to consider complex valued sequences, with small modification of the definitions.
2. Similarly to the classical Z -transforms, the transforms Ξ_k allow to convert a linear difference equation in ℓ_r^- into an algebraic equation and express the solution of this equations via the corresponding transfer function. It can be done via an appropriate extension of the equations on $t > 0$ as backward difference equations.
3. A straightforward modification leads to analogs of transforms Ξ_k for sequences from $\ell_r(0, +\infty)$.
4. Theorems 2-3 can be used for prediction of the sequences for times $t > 0$ based on the observations for $t \leq 0$. These theorems allow to tell non-predicable sequences $\{x(s)\}_{s=-\infty}^0$ from predictable ones from $\ell_{2,BL}^-$ or \mathcal{W} . For a given $\Omega \in [0, \pi)$, the set $\ell_{2,BL}^-(\Omega)$ is closed in ℓ_2^- . The sequences from these set allows an extrapolation for $t > 0$ such described in the proof of Lemma

1 and such that resulting two-sided sequence is band-limited. It could be natural to consider the projection of an arbitrary sequence on one of these subspaces, and accept the extrapolation of this projection as the optimal forecast, following the approach from [8, 10, 16, 17, 18].

5. It is still an open question how far are away the sufficient conditions established in Theorem 1 from the necessary conditions of band-limitness.

References

- [1] N. H. Bingham, Szegő's theorem and its probabilistic descendants. *Probability Surveys* 9, 2012, 287-324.
- [2] S. Cambanis and A.R. Soltani, Prediction of stable processes: spectral and moving average representations. *Z. Wahrsch. Verw. Gebiete* 66, no. 4, 1984, 593–612.
- [3] N.Dokuchaev, "The predictability of band-limited, high-frequency, and mixed processes in the presence of ideal low-pass filters," *Journal of Physics A: Mathematical and Theoretical* **41**, No 38, 382002, 2008. (7pp).
- [4] N. Dokuchaev, "Predictability on finite horizon for processes with exponential decrease of energy on higher frequencies", *Signal Processing* **90**, Iss. 2, 2010, pp. 696–701.
- [5] N. Dokuchaev, "Predictors for discrete time processes with energy decay on higher frequencies", *IEEE Transactions on Signal Processing* **60**, No. 11, 2012, 6027-6030.
- [6] N.Dokuchaev, "On predictors for band-limited and high-frequency time series", *Signal Processing* **92**, iss. 10, 2010, pp. 2571–2575.
- [7] N. Dokuchaev, Causal band-limited approximation and forecasting for discrete time processes. Working paper: <http://arxiv.org/abs/1208.3278>, 2012.
- [8] N. Dokuchaev, Forecasting for discrete time processes based on causal band-limited approximation. ICORES 2013. In: *Proc. 2nd International Conference on Operations Research and Enterprise Systems. Barcelona, Spain. 16-18 February, 2013*. Ed. B. Vitoriano and F.Valente. 2013, pp.81-85.
- [9] P. Duren, *Theory of H^p -Spaces*. Academic Press, New York, 1970.
- [10] M. Frank and L. Klotz, A duality method in prediction theory of multivariate stationary sequences. *Math. Nachr.* 244, 2002, 64–77.

- [11] J.J. Knab, "Interpolation of band-limited functions using the approximate prolate series", *IEEE Transactions on Information Theory* **25** (6), 1979, pp. 717–720.
- [12] A.N. Kolmogorov, Interpolation and extrapolation of stationary stochastic series. *Izv. Akad. Nauk SSSR Ser. Mat.*, 5:1, 1941, 3–14.
- [13] R.J. Lyman, W.W. Edmonson, S. McCullough, and M. Rao, "The predictability of continuous-time, bandlimited processes," *IEEE Transactions on Signal Processing* **48**, Iss. 2, 2000, pp. 311–316.
- [14] R.J. Lyman and W.W. Edmonson, "Linear prediction of bandlimited processes with flat spectral densities.," *IEEE Transactions on Signal Processing* **49**, Iss. 7, 2001, pp. 1564–1569.
- [15] F. Marvasti, "Comments on "A note on the predictability of band-limited processes,"" *Proceedings of the IEEE* **74** (11), 1986, p. 1596.
- [16] A.G. Miamee, and M. Pourahmadi, Best approximations in $L^p(d\mu)$ and prediction problems of Szegő, Kolmogorov, Yaglom, and Nakazi. *J. London Math. Soc.* 38, no. 1, 1988, 133–145.
- [17] T. Nakazi, Two problems in prediction theory. *Studia Math.* 78, no. 1, 1984, 7–14.
- [18] T. Nakazi and K. Takahashi, Prediction n units of time ahead. *Proc. Amer. Math. Soc.* 80, no. 4, 1980, 658–659.
- [19] A. Papoulis, "A note on the predictability of band-limited processes," *Proceedings of the IEEE* **73** (8), 1985, pp. 1332–1333.
- [20] B. Simon, Szegő's Theorem and Its Descendants. Spectral Theory for L^2 Perturbations of Orthogonal Polynomials. *M.B. Porter Lectures*. Princeton University Press, Princeton, 2011.
- [21] G. Szegő, Beiträge zur Theorie der Toeplitzschen Formen. *Math. Z.* 6, 1920, 167–202. Part II: *Math. Z.* 9, 1921, 167–190.
- [22] S. Verblunsky, On positive harmonic functions (second paper). *Proc. London Math. Soc.* 40, 1936, 290–320.